The (∞,1)-accidentopos model of unintentional type theory

(Extended Abstract)

Carlo Angiuli

March 20, 2013

1 Introduction

Dependent type theory associates to any elements x, y of a type A an identity type $x =_A y$, the type of proofs of equality of these elements. In PER Martin-Löf's extensional type theory, the identity type is a subsingleton inhabited precisely when x and y are judgmentally equal. Semantically, types are thus sets equipped with equivalence relations given by their identity types.

Groupoid Martin-Löf's intensional type theory (ITT), in contrast, does not explicitly prohibit this type from having other elements; Hofmann and Streicher showed in [4] that any closed intensional type A can be interpreted as a groupoid $[\![A]\!]$, where terms $x, y : A$ are objects $[\![x]\!], [\![y]\!] \in [\![A]\!]$, and $[\![x =_A y]\!]$ is the discrete groupoid $\hom_{[\![A]\!]}([\![x]\!], [\![y]\!])$.

In general, the identity type itself can have non-trivial morphisms, resulting in an infinite tower of non-trivial identity types. This observation has given rise to the Homotopy Type Theory project [1], which has provided new semantics of Infinity-Groupoid Martin-Löf's intensional type theory in simplicial sets [5], or globular strict [8] or weak ∞-groupoids.

This work explores the lesser-known *unintentional type theory* (UTT), which has a *mistaken identity* type $x{:}A \overset{?}{\approx} y{:}B$ of inadvertent conflations of the terms $x : A$ and $y : B$. The mistaken identity type greatly increases the expressive power of UTT by internalizing many proofs which previously required metametatheoretic techniques (e.g., user error on a blackboard).

This abstract proceeds as follows: In section 2, we discuss a number of similar logics, and the relationship between the homotopy type theory project and UTT. In section 3, we review the rules of UTT. In section 4, we resolve affirmatively the conjecture that UTT is an internal language of (∞,1)-accidentoposes.

2 Related Work

The mistaken identity type can be seen as a generalization of the handwaving and drunken modalities described by Simmons [6]. The primary difference is that all UTT judgments are handwaving judgments in Simmons's sense, since one can never be certain that important details have not been handwaved away. (The drunken modality is not expressible directly in UTT, though it frequently leads to inhabitants of the mistaken identity type.)

UTT is similar in strength to Falso [2], although UTT is a constructive logic.

The Univalent Foundations project has successfully used ITT as a "natively homotopical" language for proving theorems about spaces [7]. As in homotopy type theory, UTT has an infinite tower of iterated mistaken identity types representing the compounding nature of errors. We expect that corresponding results should be provable in UTT, such as the Freudenthal suspension-of-disbelief theorem (that, within a certain range of plausibility, it is possible to convince oneself of dubious results).

3 Syntax

Most of the rules of unintentional type theory are identical to those of ordinary type theory, as in [3].

Given any two terms, it is possible to inadvertently conflate them.

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash (M{:}A \overset{?}{\approx} N{:}B)\ \mathsf{type}} \overset{?}{\approx} F$$

Given any reason to conflate two terms, they can be inadvertently conflated; notice, however, that the original reason is subsequently forgotten in the proof.

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B \quad \Gamma \vdash \textbf{FIXME:}\ \text{are these equal?}}{\Gamma \vdash (\cancel{M{:}A}N{:}B) : (M{:}A \overset{?}{\approx} N{:}B)} \overset{?}{\approx} I$$

Lastly, given a mistaken conflation between two terms, the J eliminator allows us to replace the former term by the latter anywhere inside another term P whose type may depend on the former.

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash p : (M{:}A \overset{?}{\approx} N{:}B) \quad \Gamma, x : A \vdash B\ \mathsf{type} \quad \Gamma \vdash P[M/x] : B[M/x]}{\Gamma \vdash \text{(╯°□°) ╯︵(Ԁ·}x\text{‘}d) : B[N/x]} \overset{?}{\approx} E$$

$$\text{(╯°□°) ╯︵(Ԁ·}x\text{‘}d) \equiv P[N/x]$$

Given a mistaken identity across different types, this eliminator computes to an ill-typed term; even when $A = B$, it can easily result in a contradiction. In fact, mistaken identities can quickly propagate through arbitrary types without leaving any trace in the proof term. This is intended behavior, since a UTT judgment $\Gamma \vdash M : A$ as the assertion that, as far as the author can tell, M ought to have type A in Γ.[1]

[1] In UTT, any evidence to the contrary is just, like, your opinion, man.

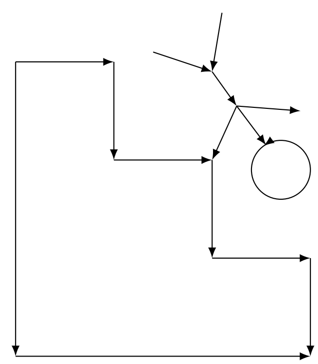
4 Semantics

An $(\infty, 1)$-accidentopos is a higher-dimensional analogue of a 1-accidentopos, a category which behaves like the category of sheeshes on a spacing-out (i.e., a category whose objects are frustrations that an inadvertent mistake has been made, and whose morphisms are transformations from these frustrations to error correction).

The $(\infty, 1)$-accidentopos model of UTT has as objects all globular sets which could be confused with an ∞-groupoid, and as morphisms all likely functors between them.

As usual, contexts are modeled as objects, dependent types as fibrations, terms as sections of those fibrations, and uncaught errors as retractions of papers. Naturally, mistaken identities are modeled by object misclassifiers.

We only sketch the proof of the descent condition.



Acknowledgements

Thanks to Chris Martens for suggesting that I study UTT, and the Univalent Foundations program for making it seem like a wise idea.

References

[1] Homotopy type theory website. `http://www.homotopytypetheory.org`, 2011.

[2] Amarilli, A. Falso. `http://www.eleves.ens.fr/home/amarilli/falso/`.

[3] Hofmann, M. Syntax and semantics of dependent types. In *Semantics and Logics of Computation* (1997), Cambridge University Press, pp. 79–130.

[4] Hofmann, M., and Streicher, T. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory* (1998), Oxford University Press.

[5] Kapulkin, C., Lumsdaine, P. L., and Voevodsky, V. The simplicial model of univalent foundations. `http://arxiv.org/abs/1211.2851`, Nov. 2012.

[6] Simmons, R. J. A non-judgmental reconstruction of drunken logic. In *The 6th Biarennial Workshop about Symposium on Robot Dance Party of Conference in Celebration of Harry Q. Bovik's 0x40th Birthday* (2007).

[7] Univalent Foundations Program. The HoTT Book. `http://github.com/hott/book`, in progress.

[8] Warren, M. The strict ω-groupoid interpretation of type theory. In *Models, Logics and Higher-Dimensional Categories* (2011), CRM Proc. Lecture Notes 53, Amer. Math. Soc., pp. 291–340.